Updraft/Downdraft Constraints for Moist Baroclinic Modes and Their Implications for the Short-Wave Cutoff and Maximum Growth Rate

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ABSTRACT

This paper examines the dynamics of moist baroclinic modes, based on the idealized model of moist baroclinic instability devised by Emanuel et al. These authors found that the finite static stability along the downdraft prevents the explosive short-wave cyclogenesis of the zero stratification limit in the moist problem, and allows only moderate (order 2) changes in the growth rate and short-wave cutoff, even when the moist static stability vanishes. To understand the limiting role of the dry static stability, a constraint is derived in this paper that relates the updraft and downdraft structures. This constraint is based on continuity and implies that a bulk wavenumber (defined in the paper) scales as the relevant deformation radius in each region.

Because neutral solutions are separable, the vertical structure can be encapsulated in terms of a single, equivalent wavenumber based on the downdraft width. This allows an interpretation of the results in terms of the equivalent dry mode. As the ratio between moist and dry static stability decreases, the downdraft width takes an increasingly larger fraction of the total wavelength. In the limit of moist neutrality all the wavelength is occupied by the downdraft, so that the short-wave cutoff is halved.

The vertical phase tilt makes unstable solutions nonseparable, and prevents defining an equivalent wavenumber in that case. However, the constraint between the bulk wavenumbers still applies. As the moist stability is reduced, the updraft solution becomes more suboptimal; in the limit of moist neutrality, the updraft wavenumber equals the short-wave cutoff. This provides a bound to the maximum growth rate in the moist problem, which is in agreement with the results of Emanuel et al.

1. Introduction

Despite the remarkable success of present-day numerical weather prediction models, the question of how moisture affects baroclinic instability remains largely unresolved at the conceptual level. Only a few studies have approached this problem at a level of simplicity that allows a basic understanding comparable, for instance, to the classical works in baroclinic instability (Eady 1949; Charney 1947). As a result, our ever-improving comprehensive numerical simulations have not been accompanied by a parallel improvement in the understanding of how moisture affects baroclinic instability. We lack in particular a simple way to assess the instability of a moist basic state, something like the Eady growth rate (Lindzen and Farrell 1980) in the dry problem. This question is especially relevant for the global warming debate, as the hydrological cycle should accelerate in a warmer world.

From the traditional perspective of the energy cycle, we can understand the destabilizing effect of moisture in terms of the diabatic generation of eddy available potential energy due to the latent heat release along the warm parts of the wave. The difficulty, however, is that this generation depends on where the precipitation occurs (e.g., Pavan et al. 1999). The simplest approach is perhaps that pioneered by Emanuel et al. (1987, hereafter EFT) in a two-level semigeostrophic Eady framework. These authors assumed that precipitation only occurs along the updrafts, which are also always saturated. With these assumptions the moist baroclinic instability problem can be solved without carrying an explicit moisture equation, and thus interpreted using

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only dry concepts. The primary effect of this precipitation pattern is a reduced static stability along the updrafts, due to the latent heat release. EFT found that for vanishing updraft stability, as is observed following slantwise convective adjustment (Emanuel 1988), the updrafts become much narrower than the downdrafts and the wavelength of the short-wave cutoff is halved. However, even in that limit the destabilizing effect of moisture is moderate: a growth rate amplification of roughly 2.5. Subsequent studies using the same parameterization found broadly consistent results, for instance by Joly and Thorpe (1989) with enhanced vertical resolution and by Fantini (1990, 1999) including nongeostrophic effects. The latter work also addressed the effect of three-dimensionality, showing that the contraction of the updraft only occurs zonally in the 3D problem. Finally, Whitaker and Davis (1994) considered the more realistic scenario of a height-dependent moist static stability, in which case the short-wave cutoff disappears and the growth rate decreases.

All these studies suggest that the inclusion of moisture has a moderate impact on the baroclinic growth rate, even in the limit of moist neutrality. Since the short-wave growth rate becomes infinite for zero stratification, it must be the dry static stability along the downdraft that ultimately prevents the explosive growth in that limit. This seems reasonable, as we expect the modes to feel both the dry and moist stability. However, it is unclear from previous studies how the stability along each region constrains the mode as a whole. This paper provides insight into this issue by means of some fundamental constraints relating the updraft and downdraft solutions, which help clarify certain dynamical aspects of EFT's results (we emphasize that no new numerical result is provided). Section 2 discusses the continuous 2D problem, and shows that neutral solutions are separable and can be characterized in terms of an equivalent static stability. In contrast, unstable solutions are nonseparable because of the different phase tilt along updrafts and downdrafts. Section 3 shows this to be a consequence of the different character of the solution along each region, and exploits this insight in a new asymptotic expression for the growth rate in agreement with EFT's eigenvalue solution. Finally, section 4 discusses the implication for the growth rate sensitivity to changes in the dry/moist stability, and in turn for the eddies in a warmer climate.

2. The vertical structure in the continuous problem

In contrast to EFT and most other subsequent studies we shall use quasigeostrophic (qg) theory below, as we are more concerned with the large-scale dynamical aspects than with the frontogenesis problem that motivated many previous studies. Although most of the ideas discussed have a direct semigeostrophic equivalent, we see little advantage in following that path because the only real difference between both frameworks arises from the transformation between physical and geostrophic space, which is dependent on wave amplitude and therefore arbitrary for the linear 2D problem (EFT; Moore and Montgomery 2004). The use of the qg framework, or even the geostrophic momentum approximation (Emanuel 1985), is questionable in the presence of small effective static stability along the updrafts, which leads to locally large ageostrophic velocities and accelerations [see also Fantini (1995)]. In that limit, the updraft dynamics is likely controlled by small-scale turbulence rather than by the large-scale considerations discussed here. However, this may be not so critical, as one of the key results of this study is the relatively minor role played by the updraft.

With this caveat, consider a two-dimensional Boussinesq fluid on an f plane. Under quasigeostrophic scaling, the linearized vorticity equation about a basic state $U(z)$ is given by

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi_{xx} - f_0 w_z = 0. \qquad (1)$$

In the adiabatic limit, potential temperature is conserved for unsaturated motion, and equivalent potential temperature for saturated motion (Emanuel 1994). EFT show that both can be combined in a single thermodynamic equation [see also Fantini (1995)]

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi_z - U_z\phi_x + wS = 0, \qquad (2)$$

provided that a different stability S is used over the unsaturated regions ($S = S_d = N^2/f_0$) and the saturated regions ($S = S_m = rN^2/f_0$). Following EFT, we assume the updrafts to be always saturated, and the downdrafts to be always unsaturated, so that S_m (S_d) represents the effective static stability along the updraft (downdraft).[1] The parameter $r < 1$, a function of the dry and moist adiabatic lapse rates, is defined in Fantini (1995).

We assume a modal time-evolution $(\partial/\partial t) \sim \sigma$, and consider an Eady basic state with depth H and zonal wind $U = U_0 + U_s \times (z/H) = U_s \times (z + z_0)/H$. We nondimensionalize as follows:

$$x = \lambda^{-1}\tilde{x}, \quad z = H\tilde{z}, \quad \phi = (U_s\lambda^{-1})\tilde{\phi},$$

$$w = (U_s^2 S_d^{-1} H^{-1})\tilde{w},$$

[1] In the following, the subscripts m/d are used to refer to a moist moist/dry parameter, while the subscripts u/d are used to refer to the updraft/downdraft.

where $\lambda = (f_0/S_d)^{1/2}H^{-1}$ is the inverse Rossby radius (based on the dry stability).

We also eliminate $\tilde{\phi}$ in favor of $\tilde{w}$ (using continuity of $\tilde{w}$, $\tilde{w}\tilde{S}$ when differentiating) to get

$$\left(\tilde{\sigma} + (\tilde{z} + \tilde{z}_0)\frac{\partial}{\partial \tilde{x}}\right)(\tilde{w}_{\tilde{z}\tilde{z}} + \tilde{w}_{\tilde{x}\tilde{x}}\tilde{S}) - 2\tilde{w}_{\tilde{x}\tilde{z}} = 0, \quad (3)$$

where $\tilde{\sigma} = (\sigma/U_s\lambda)$ and $\tilde{S}$ adopts a different value along downdrafts ($\tilde{S} = 1$) and updrafts ($\tilde{S} = r$). Hereafter, the vertical coordinate $\tilde{z}$ is redefined with origin at $-\tilde{z}_0$ and tildes are dropped. Finally, we exploit the Galilean invariance of the problem and choose the surface wind z_0 to render the solution stationary and to make σ real. This entails reformulating the problem as an eigenvalue problem for z_0 rather than for c_r, as is more traditional.

Consider the neutral limit $\sigma = 0$ first. In that case, Eq. (3) can be integrated in x once to yield the separable equation

$$w_{zz} - 2\frac{w_z}{z} + w_{xx}S = 0, \quad (4)$$

Rewriting $w(x, z) = X(x)\,Z(z)$

$$\frac{Z''}{Z} - \frac{2}{z}\frac{Z'}{Z} = -S\frac{X''}{X} = \alpha^2, \quad (5)$$

where α^2 is a separation constant that may be regarded as an equivalent wavenumber.

The horizontal structure $SX''(x) + \alpha^2 X = 0$ consists of simple sinusoidal solutions, with wavenumbers $k_d = \alpha$ and $k_u = \alpha r^{-1/2}$ along the downdraft and updraft, respectively. We construct the full solution defining the origin $x = 0$ at the western side of the updraft

$$X(x) = X_u \sin(\alpha r^{-1/2}x) \quad x < L_u = \frac{\pi}{\alpha r^{-1/2}}, \quad (6)$$

$$X(x) = -X_d \sin[\alpha(x - L_u)] \quad 0 < x - L_u < L_d = \frac{\pi}{\alpha}. \quad (7)$$

Thus, in each region the local wavenumber scales as the dry/moist Rossby radius, so that the ratio between the updraft and downdraft widths is given by the simple analytical expression $L_u/L_d = r^{1/2}$. We can also derive the following quantization condition:

$$\alpha = \frac{(1 + r^{1/2})\pi}{L} = \frac{\pi}{L_d}, \quad (8)$$

where $L = L_u + L_d$ is the (dimensionless) wavelength of the mode.

When the temperature perturbation ϕ_z is continuous across the updraft/downdraft front, Eq. (2) implies that $\int wS\,dx = 0$ in the neutral case. For $r \neq 1$, this condition is incompatible with mass continuity $\int w\,dx = 0$, implying that either ϕ_z or the ageostrophic flow u_a must have a jump across the front. In the former case, one gets a downdraft to updraft ratio $X_d/X_u = r^{1/2}$, and in the latter $X_d/X_u = r^{3/2}$.

On the other hand, the vertical structure equation: $Z'' - 2Z'/z - \alpha^2 Z = 0$ does not depend on the value of S, or on whether S is constant or not, which is of course a consequence of separability. Thus, the vertical structure of the solution is the same as in the dry Eady problem with $r = 1$ and the same value of α. It is easy to check that the vertical velocity eigenmodes of the dry Eady problem $Z_{\pm}(z) = (\pm\alpha z - 1)\exp(\pm\alpha z)$ satisfy that equation.

Hence r only affects the vertical structure of the modes through the eigenvalue $\alpha(r)$. The quantization condition (8) defines an equivalent wavenumber for the moist problem, such that the vertical structure of a moist mode agrees with that of the dry mode with wavenumber $\alpha(r)$. Equation (8) essentially implies that the Rossby depth of a moist mode is controlled by the mean static stability $(S_d^{1/2} + S_m^{1/2})/2$ between updrafts and downdrafts.

Note that moist modes are always deeper than the dry mode of the same length. As r decreases, the downdraft occupies an increasingly larger fraction of the total length, so that in the limit $r \to 0$, all the wavelength is occupied by the downdraft. In that limit, the Rossby depth of the mode is twice that of the dry mode with the same total wavelength. This explains EFT's results that the short-wave cutoff does not depend on r for a fixed downdraft width, and is just halved in the $r = 0$ limit for a fixed total length.

It only remains to calculate z_0 (or equivalently, the phase speed of the modes). To do so, we note that only for certain values of z_0 a linear combination of the vertical velocity eigenmodes $Z(z) = A \times Z_+(z + z_0) + B \times Z_-(z + z_0)$ can be forced to satisfy the boundary conditions $Z(0) = Z(1) = 0$ with nonzero values of the constants A, B. This defines z_0 as the eigenvalue of the problem when L is given. We can find the constraint $z_0(L; r)$ without explicitly solving this problem by noting that U_0 must be such as to render the dry Eady problem with equivalent wavenumber α stationary: $z_0 = -c_{r,\mathrm{dry}}(\alpha)$, where $c_{r,\mathrm{dry}}(k)$ is the dispersion relation of the neutral dry Eady mode and $\alpha(L; r)$ is defined by Eq. (8).

In the unstable case, Eq. (3) shows that the problem is no longer separable when $\sigma \neq 0$. The lack of separability is a consequence of the vertical phase tilt with height characteristic of baroclinic instability, which arises even in the dry problem. Nevertheless, in that case a change of variable exists $x' = x - \Theta(z)$, $z' = z$

that makes the solution separable in tilted x', z' coordinates [$\Theta(z)$ is the phase tilt of the modes, and should be determined as part of the solution]. However, this procedure cannot work in the moist case because the vertical phase tilt is different along updrafts and downdrafts. For instance, the numerical simulations of Joly and Thorpe (1989) show that when r is small, the tilt is much smaller along the updraft than along the downdraft regions (cf. their Fig. 9). Physically, only the downdrafts grow through baroclinic processes, while the updraft growth occurs through latent heat release. Equivalently, the solution is strongly suboptimal along the updraft, in the sense that it grows much slower than the most unstable mode with constant static stability the moist stability $S = S_m$. This will be discussed in detail in the next section.

3. One-dimensional (two layer) model

As discussed in the previous section, a significant complication in the unstable case is the phase tilt of the modes, which makes the solution nonseparable. EFT did not encounter this difficulty because in the two-layer model the vertical velocity is only defined at midlevel, which makes the problem effectively one-dimensional. We also take advantage of this simplification, and restrict the analysis of the unstable problem below to the two-layer case. Our formulation closely follows EFT, except that we use a quasigeostrophic model.

The inviscid linearized equations for this model can be written in terms of the barotropic [$\phi = (\psi_1 + \psi_2)/2$] and baroclinic [$\tau = (\psi_1 - \psi_2)/2$] streamfunctions as follows:

$$\sigma\phi_{xx} + U\tau_{xxx} = 0, \tag{9}$$

$$\sigma\tau_{xx} + U\phi_{xxx} + w\frac{f_0}{H} = 0, \tag{10}$$

$$\sigma\tau - U\phi_x + wSH = 0. \tag{11}$$

The first two equations are the barotropic and baroclinic vorticity equations, and the third one is the thermodynamic equation. The basic flow has zonal velocity U ($-U$) in the upper (lower) layer. Also S is a stability parameter proportional to the stratification, and H the depth of the equal layers. We assume a temporal evolution of the form $(\partial/\partial t) \sim \sigma$. As in the previous section, we require σ to be real, which is only possible for certain values of the surface wind. The vertical symmetry of the problem implies that, as in the dry case (Holton 1992), the steering level is at midlevel for all unstable solutions so that $z_0 = 0$. This is also consistent with EFT's results that moist modes are stationary when the zonal wind is purely baroclinic.

The set of Eqs. (9)–(11) is independently solved along the updraft and downdraft regions. Along the updraft, $S = S_m$, while along the downdraft $S = S_d$. Then both solutions are coupled across the lines (or rather, the points) of zero w. The boundary conditions enforce the continuity of the ageostrophic circulation, pressure and temperature across those lines. Combining Eqs. (9)–(11), we eliminate all variables in favor of w to get

$$\frac{1}{\lambda^4}\frac{\partial^4 w}{\partial x^4} + \left(1 - \frac{\sigma^2}{U^2\lambda^2}\right)\frac{1}{\lambda^2}\frac{\partial^2 w}{\partial x^2} + \frac{\sigma^2}{U^2\lambda^2}w = 0. \tag{12}$$

This equation admits exponential solutions e^{ikx} with wavenumber k given by

$$\frac{k^2}{\lambda^2} = \frac{1 - g^2 \pm \sqrt{(1 - g^2)^2 - 4g^2}}{2}, \tag{13}$$

where $g = (\sigma/U\lambda)$ is the dimensionless growth rate. Of course, this equation is just the inverse of the dispersion relation (Holton 1992)

$$\frac{\sigma}{U\lambda} = \frac{k}{\lambda}\left(\frac{\lambda^2 - k^2}{\lambda^2 + k^2}\right)^{1/2}. \tag{14}$$

Note that there are always two values of k^2/λ^2 giving the same growth rate in Eq. (13), consistent with the fact that the dispersion relation has a maximum and is multivalued. The maximum growth rate of the dry problem $g_{max} = \sqrt{2} - 1$ is such that the argument of the square root vanishes in Eq. (13). Larger values of g require a complex wavenumber, which cannot satisfy the periodic boundary conditions.

While in the neutral case $g = 0$, Eq. (13) gives $k_u/\lambda_m = k_d/\lambda_d$ as discussed in the previous section, this is no longer the case in the unstable problem. For modal solutions σ must be the same along updraft and downdraft, but the dimensionless growth rate g is generally different: $g_u/g_d = \lambda_d/\lambda_m = r^{1/2}$. As a result, Eq. (13) implies that k/λ is also different. It seems reasonable to assume that the maximum growth rate of the moist problem should be larger than that for the dry problem with static stability S_d, and smaller than for the dry problem with static stability S_m. This implies

$$g_u \le \sqrt{2} - 1 \le g_d,$$

that is, the updraft solution is suboptimal and the downdraft solution is superoptimal.

Hence, k_u will always be real, while k_d will be complex for the most unstable moist mode (or, more generally, for any moist mode growing faster than the most unstable mode of the dry problem). Although σ may in principle grow as fast as $r^{-1/2}$ in the limit $r \to 0$, the dimensionless updraft growth rate is still small: $g_u^2 \leq 0.17$ (for finite σ, g_u furthermore goes to zero as $r \to 0$), allowing to expand Eq. (13) in powers of g_u^2. To order g_u^2, the two roots are

$$k_{u1}^2/\lambda_m^2 \approx 1 - 2g_u^2 \quad k_{u2}^2/\lambda_m^2 \approx g_u^2 \ll 1. \tag{15}$$

The updraft solution is not very sensitive to growth rate in the small r limit. The primary wavenumber decreases slightly from the neutral case, and there is also a long-wave correction.

On the other hand, the downdraft wavenumber k_d becomes complex for any mode growing faster than the maximum dry growth rate $g_d \geq \sqrt{2} - 1$. For $g_d \leq \sqrt{2} + 1$ the two roots in Eq. (13) are complex conjugate, whereas for larger g_d both k_d^2 roots are real and negative, implying purely exponential solutions. It is easy to show that solutions of the latter form cannot satisfy the boundary conditions. Hence, g_d is bounded by the above value and the downdraft roots are of the form $\pm k_d$, $\pm k_d^*$, where k_d^* is the complex conjugate of k_d. If, following EFT, we assume the downdraft to be symmetric about its midpoint ($x = 0$), the solution is

$$\begin{aligned} w_d &= \Re\{A \cos(k_d x)\} \\ &= A_r \cos(k_{dr}x)\cosh(k_{di}x) + A_i \sin(k_{dr}x)\sinh(k_{di}x), \end{aligned} \tag{16}$$

where $\Re$ stands for the real part of the complex expression, A is a complex constant, and k_{dr}, k_{di}, A_r, A_i are the real and imaginary parts of k_d and A, respectively. For complex k_d, there is not a direct relation between k_{dr} and the wavelength as in the real case. If L_d is the downdraft width, the condition $w_d(\pm L_d/2) = 0$ implies that $A\cos(k_d L_d/2)$ is purely imaginary, which constrains the phase of A but allows a nonzero amplitude even if $k_{dr}L_d \neq \pi$.

When the solution is not a pure wave, the squared local wavenumber $-w_{xx}/w$ is a function of x. It is useful, however, to define a bulk wavenumber as follows:

$$\tilde{k} = \left(-\frac{\int w_{xx}\,dx}{\int w\,dx} \right)^{1/2}, \tag{17}$$

where $\tilde{k}$ is independently defined along updraft and downdraft, with the integrals extending over their corresponding length. It is shown in the appendix that the continuity of pressure, temperature, and the ageostrophic circulation across the lines of zero w imply that

$$\frac{\tilde{k}_d}{\lambda_d} = \frac{\tilde{k}_u}{\lambda_m}, \tag{18}$$

which may be regarded as the generalization of the neutral wavenumber law.

In the small r limit, the updraft structure is not very different from the half sine of the neutral case, so that we can approximate its bulk wavenumber as

$$\tilde{k}_u^2 \approx k_{u1}^2 \approx \lambda_m^2(1 + O(g_u^2)).$$

Along the downdraft, $\tilde{k}_d$ appears in principle a function not just of k_d, but also of A. However, we can eliminate this dependence using the boundary conditions. The fact that $A\cos(k_d L_d/2)$ is purely imaginary implies that $A\sin(k_d L_d/2)$ is a real number. Thus we can write

$$\begin{aligned} \tilde{k}_d^2 &= \frac{-w_x(L_d/2)}{\int_0^{L_d/2} w\,dx} = \frac{\Re\{A k_d \sin(k_d L_d/2)\}}{\Re\{A k_d^{-1} \sin(k_d L_d/2)\}} \\ &= \frac{\Re\{k_d\}}{\Re\{k_d^{-1}\}} = |k_d|^2. \end{aligned}$$

This result, together with Eq. (18), puts a bound on the total (real plus imaginary) wavenumber for the downdraft $|k_d|^2 = k_{dr}^2 + k_{di}^2$:

$$\frac{|k_d|^2}{\lambda_d^2} = \frac{\tilde{k}_d^2}{\lambda_d^2} = \frac{\tilde{k}_u^2}{\lambda_m^2} \approx \frac{k_u^2}{\lambda_m^2} \leq 1.$$

Note that this expression resembles the dry result that the (real) wavenumber $k_d < \lambda_d$ for instability (the short-wave cutoff). However, in the moist case the condition $|k_d| < \lambda_d$ results from the short-wave cutoff for the updraft: $k_u < \lambda_m$, rather than for the downdraft. It is only because of the law of bulk wavenumbers Eq. (18) that this condition also translates into a constraint for the downdraft wavenumber $|k_d|$.

It is this constraint that also limits the maximum dimensionless growth rate for the downdraft. To see this, we can take the modulus of Eq. (13) (note that the argument of the square root is negative in that equation for $\sqrt{2} - 1 \leq g_d \leq \sqrt{2} + 1$), which gives

$$g_d = \frac{|k_d|^2}{\lambda_d^2} \leq 1, \tag{19}$$

which is a tighter bound than $g_d < \sqrt{2} + 1$, corresponding to the transition to purely imaginary updraft wavenumbers. Equation (19) gives a maximum growth rate amplification with respect to the dry problem of

$1/(\sqrt{2} - 1) \approx 2.4$, reasonably close to EFT's numerical results.

This maximum downdraft growth rate, $g_d = 1$, is approached as $r \to 0$ and k_u asymptotes λ_m. We can also get an approximate expression for $g_d(r)$ by relating the updraft and downdraft dimensionless growth rates:

$$g_u = \frac{k_u}{\lambda_m}\left[\frac{1 - \left(\frac{k_u}{\lambda_m}\right)^2}{1 + \left(\frac{k_u}{\lambda_m}\right)^2}\right]^{1/2} = r^{1/2} g_d = r^{1/2}\frac{|k_d|^2}{\lambda_d^2}$$

$$\approx r^{1/2}\left(\frac{k_u}{\lambda_m}\right)^2.$$

Solving for $(k_u/\lambda_m)^2$ we get

$$g_d(r) \approx \left(\frac{k_u}{\lambda_m}\right)^2 = \frac{-(r+1) + \sqrt{(r+1)^2 + 4r}}{2r}. \qquad (20)$$

This expression is plotted in Fig. 1a alongside EFT's numerical results for the maximum growth rate.[2] The agreement is reasonable but not perfect, probably because of the inaccuracy of the only approximation made: $\tilde{k}_u \approx k_u$. Interestingly, Eq. (20) predicts the exact solution in the dry limit ($r = 1$, $g_d = \sqrt{2} - 1$), as there is no distinction between wavenumber and bulk wavenumber in that case. It is also noteworthy that, despite the enhanced σ, $g_u = r^{1/2} g_d$ actually decreases as $r \to 0$ (see Fig. 1b). This is because the inverse moist Rossby radius λ_m grows faster than σ as r decreases. Hence, the updraft solution becomes more and more suboptimal—and thus closer and closer to the neutral limit described in the previous section—with decreasing r. In the limit $r \to 0$, $g_u \to 0$ and $k_u \to \lambda_m$ (the short-wave cutoff).

To conclude, we note that the concepts introduced here should also apply to the continuous problem. In that case as well, the condition that the downdraft growth rate exceeds the maximum value for the dry problem implies that the downdraft wavenumber must be complex. In other words, it is the enhancement in the total downdraft wavenumber, due to the addition of an imaginary component k_{di}, that allows the growth rate to increase beyond its dry maximum for the given S_d. However, because the law of bulk wavenumbers still applies (see the appendix), the bulk downdraft wavenumber $\tilde{k}_d$ should again be bounded, which translates into a maximum growth rate. The same methodology of this section could be applied to the continuous Eady problem, though finding an analytical relation between g_d and $\tilde{k}_d$ is much more involved in that case because of the transcendental character of the dispersion relation.

[2] Note that to make EFT's results comparable to ours they had to be rescaled by a $\sqrt{2}$ factor, so as to account for differences in the nondimensionalization.

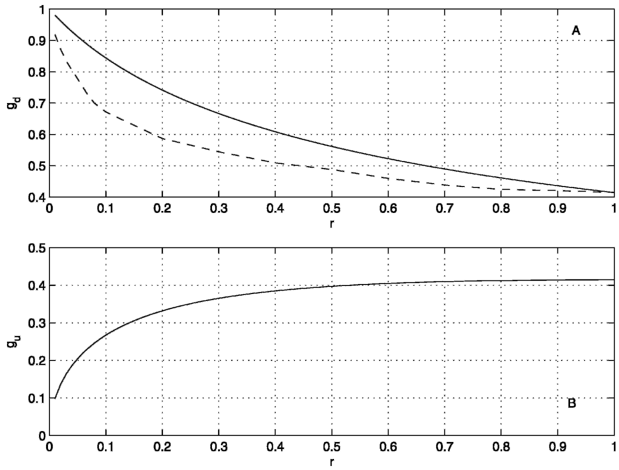

FIG. 1. (a) Downdraft-normalized growth rate (EFT's results are shown dashed); (b) updraft-normalized growth rate.

4. Discussion

An implication of the previous results is that in the small r limit, the growth rate is most sensitive to changes in the dry stability. To see this, we express the dimensional growth rate

$$\sigma \propto S_d^{-1/2} F(r) = S_d^{-1/2} F(S_u/S_d),$$

where $F(r) = (k_u/\lambda_m)^2$ is given by Eq. (20). Using this expression, we can estimate the sensitivity of the growth rate to changes in the dry or moist stability keeping the other stability constant,

$$\left(\frac{\partial \ln\sigma}{\partial \ln S_m}\right)_{S_d} = \frac{d \ln F}{d \ln r}, \qquad (21)$$

$$\left(\frac{\partial \ln\sigma}{\partial \ln S_d}\right)_{S_m} = -\frac{1}{2} - \frac{d \ln F}{d \ln r}. \qquad (22)$$

The results are plotted in Fig. 2 as a function of r (note the logarithmic scale to emphasize the small r limit). As can be seen, for small r ($r < 0.225$) the growth rate is more sensitive to changes in the dry stability than to changes in the moist stability, while the reverse is true for large r. The reason is that for small r the corrective factor $F(r)$ asymptotes 1 and is little sensitive to further reductions in r. This is illustrated in Fig. 3, which shows how the growth rate (as a function of r) is affected when either the dry or moist stability is reduced by a 50%, keeping the other stability constant (which implies changing r). It is obvious that changes in the dry stability have the largest impact, particularly when r is small.

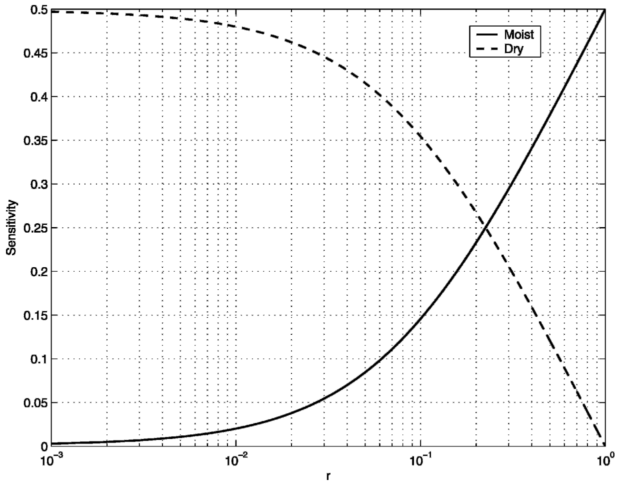

FIG. 2. Sensitivity of the growth rate of the mode to changes in either the moist (solid) or dry (dashed) static stability, as a function of *r*.

This stronger sensitivity to the dry static stability may have climatic implications. An open question in the global warming debate is whether baroclinic eddies would be weaker or stronger in a warmer climate. Contrasting arguments can be provided that support one or another hypothesis [see Held (1993) for an illuminating discussion]. On the one hand, the enhanced latent heat transport in a warmer climate makes the eddies more efficient, so that weaker eddies would be needed to transport the same amount of heat (or, if the net transport increases, the reduction in the temperature gradient should lead to a weakening of the eddies). On the other hand, latent heat is another source of energy for the eddies, and moist eddies grow faster than dry ones. This at least opens the possibility that the enhanced moistening could lead to stronger eddies in a warmer climate. Both scenarios can also be compatible if the eddies grow faster but are either less frequent or shorter-lived (e.g., Branscome and Gutowski 1992; Gutowski et al. 1992).

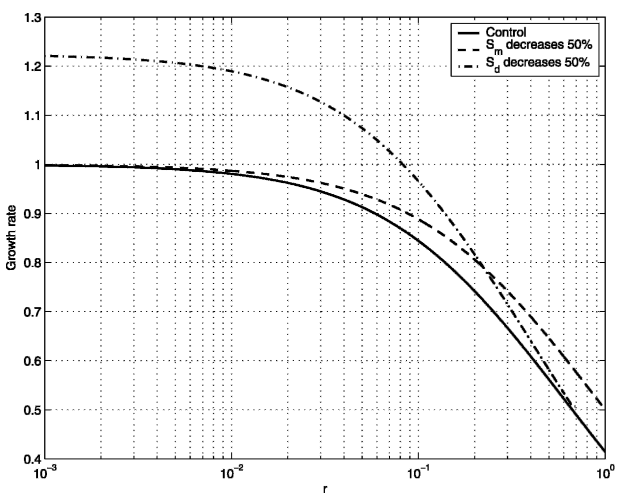

FIG. 3. Change in the growth rate of the mode when either the dry (dash–dotted) or moist (dashed) static stability is reduced by 50%. The *r*-dependent reference growth rate is shown with solid line.

Yet one of the main readings of our results is that any destabilizing effect of moistening should be moderate when the atmosphere has small neutrality to saturated ascents, in which case the sensitivity to changes in the dry stability is higher than it is to changes in the moist stability. This suggests that the destabilizing effect of moistening cannot be assessed without a careful consideration of how the dry static stability changes in response to the moistening. The determination of the dry static stability is one of the fundamental questions in the general circulation of the atmosphere. The dominant view is that it is determined by the competition between the radiative destabilization and the stabilizing eddy fluxes in the midlatitudes, while convection is important in the Tropics (Held 1982; Schneider 2004). However, the static stability also varies on synoptic spatial and temporal scales, being nearly neutral for saturated ascents along the warm sectors of midlatitudes cyclones (Emanuel 1988). It has been recently argued by Juckes (2000) that by constraining the minimum static stability to be moist-neutral, convection also plays a major role in the determination of the climatological midlatitude static stability. Juckes (2000) also emphasizes the impact that as a result moisture has for the determination of the mean static stability. When the atmosphere is forced to have a fixed moist stability (neutral or otherwise), the dry stability must increase with the amount of water vapor. From that point of view, the moistening should rather lead to a weakening than an intensification of the eddies, which is consistent with the findings of Frierson et al. (2005, manuscript submitted to *J. Atmos. Sci.*) in an idealized moist GCM.

Nevertheless, we should be cautious when inferring any conclusions from the idealized model, since other factors such as water vapor availability (Pavan et al. 1999) and the interaction with convection (Jiang and Gutowski 2000; Gutowski and Jiang 1998) may also play a role in a more realistic setting. In addition, it is not clear that the modal growth rate provides a reasonable assessment of the mean state instability, as the moist problem is intrinsically nonlinear. In fact, it is not even possible to expand the general moist solution in terms of these modes, which do not form a complete set. Even in this idealized setting, the association between the updraft short-wave cutoff and the maximum growth rate opens the possibility of a more pronounced destabilizing effect on the beta plane, when the short-wave cutoff disappears (Green 1960). That could perhaps explain the turbulent simulations of Lapeyre and Held (2004), who found a stronger increase in eddy kinetic energy (EKE) with moisture than might have been expected based on the results presented here. We are unaware of any previous study that has examined this issue. When the moist stability is height dependent $r = r(z)$, there is also an implicit interior PV gradient due to the vertical gradient in heating (Montgomery and Farrell 1991; Moore and Montgomery 2004). As expected, the short-wave cutoff disappears but the growth rate also decreases (Whitaker and Davis 1994; Moore and Montgomery 2004). However, this could indicate that the equivalent moist stability along the updraft is not small.

Lifting the f-plane restriction is just one of the many ways in which this work could be extended to bridge the gap with more comprehensive moist GCMs, as only a small fraction of the idealized studies in the literature have incorporated moist effects. Additional research at the theoretical level is needed to elucidate the extent to which dry theories of the extratropical circulation are also applicable in the moist case.

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APPENDIX

Proof of the Bulk Wavenumber Law

In section 3 we defined the bulk wavenumber $\tilde{k}$

$$\tilde{k}^2 = -\frac{\int_{-L/2}^{L/2} w_{xx}\,dx}{\int_{-L/2}^{L/2} w\,dx} = -\frac{2w_x(L/2)}{\int_{-L/2}^{L/2} w\,dx}$$

for updraft and downdraft. Both updraft and downdraft are assumed to be symmetric about their midpoint $x = 0$ (note that a different, local coordinate system is used in each region) and to have widths L_u and L_d, respectively.

To show that $\tilde{k}_d/\lambda_d = \tilde{k}_u/\lambda_m$, we use the boundary conditions. These are the continuity of ϕ, τ along the updraft–downdraft fronts (where $w = 0$), as well as mass continuity

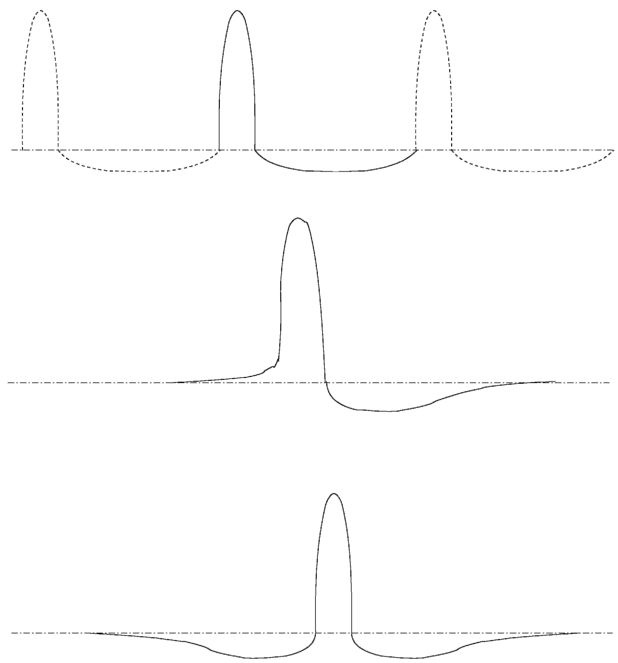

FIG. A1. Examples of updraft/downdraft configurations for which the bulk wavenumber law applies: (top) fully periodic, as in the text; (middle) updraft and downdraft vanish at infinity; (bottom) two downdraft lobes surround a single updraft.

$$\int_{-L_d/2}^{L_d/2} w\,dx = -\int_{-L_u/2}^{L_u/2} w\,dx. \tag{A1}$$

Referring to Eqs. (9)–(11), both w and wS are continuous across the fronts. The continuity of τ and wS implies through Eq. (11) that of ϕ_x. The continuity of ϕ and ϕ_x then implies through Eq. (9) that of τ_x and τ_{xx}. Finally, the continuity of w and τ_{xx} implies through Eq. (10) that of ϕ_{xxx}, which in turn implies through Eq. (9) the continuity of τ_{xxx} and τ_{xxxx}. However, note that because w_x is not continuous, ϕ_{xxxx}, τ_{xxxxx}, and all their higher order derivatives must have a jump at the front. Differentiating Eq. (11) twice, the continuity of these ϕ and τ derivatives requires that Sw_x and Sw_{xx} also be continuous across the front

$$S_m w_x(L_u/2) = S_d w_x(-L_d/2) = -S_d w_x(L_d/2), \tag{A2}$$

$$S_m w_{xx}(L_u/2) = S_d w_{xx}(-L_d/2) = S_d w_{xx}(L_d/2). \tag{A3}$$

Using these conditions together with (A1) and the definition of $\tilde{k}$, we finally get

$$\frac{\tilde{k}_d^2}{\tilde{k}_u^2} = \frac{w_x(L_d/2)}{\int_0^{L_d/2} w\,dx}\,\frac{\int_0^{L_u/2} w\,dx}{w_x(L_u/2)} = \frac{S_m}{S_d} = \frac{\lambda_d^2}{\lambda_m^2},$$

and hence that $\tilde{k}_d/\lambda_d = \tilde{k}_u/\lambda_m$, as postulated. Neglecting the antisymmetric part of the solution (unavoidable for such geometries), it is easy to extend the previous analysis to show that this constraint also holds for the symmetric part of the solution in other relevant updraft/downdraft configurations in which the vertical velocity perturbations vanish at infinity (Fig. A1), and not just for the periodic case considered here.

Finally, note that the bulk wavenumber law also applies in the continuous problem at every level, provided that the ageostrophic circulation and Sw_x are still continuous (which requires continuous θ_x and $\xi = \phi_{xx}$). As discussed in section 2, the unstable 2D problem is nonseparable, which implies that k_d should be a function of height in the continuous problem. However, the previous constraint and the fact that $\tilde{k}_u \approx k_u \neq f(z)$ in the small r limit suggest that the bulk downdraft wavenumber $\tilde{k}_d$ should still be nearly constant at all heights. For solutions of the form $w_d = \Re\{A \cos k_d x\}$, that would be the case if only the phase but not the amplitude of k_d was dependent on height.

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